Sheaf Cohomology

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Given a short exact seq of sheaves on a scheme

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

The induced map $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ need not be surjective.

Can we systematically measure the failure of this right exactness? Sheaf cohomology theory provides an answer. Indeed, we will attach functors $\{H^i(X, -)\}_{i \in \mathbb{N}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod}$ such that the above exact seq will produce an exact seq

$$0 \to \underset{\Gamma(X,\mathcal{F}')}{\underset{\shortparallel}{H^0(X, \mathcal{F}')}} \to \underset{\Gamma(X,\mathcal{F})}{\underset{\shortparallel}{H^0(X, \mathcal{F})}} \to \underset{\Gamma(X,\mathcal{F}'')}{\underset{\shortparallel}{H^0(X, \mathcal{F}'')}} \to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'')$$

$$\vdots$$

i.e the failure of surjectivity $\left(= \dfrac{H^0(X, \mathcal{F}'')}{\mathrm{Im}(X, \mathcal{F})}\right)$ is the kernel of the map $H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F})$ and so on.

Instead of working with the right exactness of the special functor $\Gamma(X, -)$, we consider the same problem for more general functors and device an analogous solution.

<u>**Convention:**</u> Functors are between abelian categories, all functors are additive.

<u>**Eg:**</u> $\Gamma : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X(X)}$, For a map of ringed spaces or schemes $f : X \to Y$, $f_* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$; $f_* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$, For $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$.

<u>§ Generalities:</u>

<u>Lemma.</u> Effaceable δ-functors are universal.

<u>Pf.</u> Let $\{T^i\}_{i \in \mathbb{N}}$ be an effaceable δ-functor $\mathcal{A} \to \mathcal{B}$. Given another δ-functor $\{S^i\}_{i \in \mathbb{N}}$ and a natural transformation $f : T^0 \to S^0$, construct $f^n : T^n \to S^n$ s.t $f^0 = f$ by induction on n.

Set $f^0 = f$, Suppose $f^1, \dots, f^n$ are already constructed.

Given $A \in \mathcal{A}$, fix an exact seq

$$0 \to A \to I \to I'' \to 0 \quad \text{s.t. } F^{n+1}(A) \to F^{n+1}(I) \text{ is}$$

the zero map.

Have a diag

$$\begin{tikzcd}
\to T^n(A) \arrow[r] \arrow[d, "f^n(A)"] & T^n(I) \arrow[r] \arrow[d, "f^n_{(I)}"] & T^n(I'') \arrow[r] \arrow[d, "f^n_{(I'')}"] & T^{n+1}(A) \arrow[r, "0"] \arrow[d, dashed, "?"] & T^{n+1}(I) \\
\to S^n(A) \arrow[r] & S^n(I) \arrow[r] & S^n(I'') \arrow[r] & S^{n+1}(A) \arrow[r] & S^{n+1}(I)
\end{tikzcd}$$

$$\begin{tikzcd}
T^{n+1}(A) = \mathrm{coker}\,(T^n(I) \to T^n(I'')) \arrow[d] \arrow[dr, bend right, "f^{n+1}(A) :="'] & \\
\mathrm{coker}\,(S^n(I) \to S^n(I'')) = \mathrm{Ker}\,(S^{n+1}(A) \to S^{n+1}(I)) \arrow[r, hook] & S^{n+1}(A)
\end{tikzcd}$$

We need to check

(i) That $f^{n+1}(A)$ does not depend on the choice of $0 \to A \to I$

(ii) functoriality, given $A \to A'$, have

$$\begin{tikzcd}
T^{n+1}(A) \arrow[r, "f^{n+1}(A)"'] \arrow[d] & S^{n+1}(A) \arrow[d] \\
T^{n+1}(A') \arrow[r, "f^{n+1}(A')"'] & S^{n+1}(A')
\end{tikzcd}$$

commutative.

(i) Given $0 \to A \to I$, $\quad 0 \to A \to I'$

consider $\quad 0 \to A \to I \oplus I' \quad$ (diag)

Have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}", "\Vert"'] & I \arrow[r] \arrow[d, shift right] & I/A \arrow[r] \arrow[d, shift right] & 0 \\
0 \arrow[r] & A \arrow[r] & I \oplus I' \arrow[r] \arrow[u, shift right] & I \oplus I'/A \arrow[r] \arrow[u, shift right] & 0
\end{tikzcd}$$

This gives

$$\begin{tikzcd}
T^n(A) \arrow[r] \arrow[d, "\mathrm{id}"] & T^n(I) \arrow[r] \arrow[d] & T^n(I/A) \arrow[r] \arrow[d] & T^{n+1}(A) \arrow[d, "\mathrm{id}"] \\
T^n(A) \arrow[r] & T^n(I \oplus I') \arrow[r] & T^n(I \oplus I'/A) \arrow[r] & T^{n+1}(A)
\end{tikzcd}$$

showing the extensions given by $0 \to A \to I$ and $0 \to A \to I \oplus I'$ are the same.

Now compare $0 \to A \to I'$ and $0 \to A \to I \oplus I'$.

(iii) Given $0 \to A \xrightarrow{\alpha} I_A$, $0 \to B \xrightarrow{\beta} I_B$, and $g : A \to B$, replace $0 \to A \to I_A$ by

$$0 \to A \xrightarrow{(\alpha,\ \beta\cdot g)} I_A \oplus I_B, \quad \text{Then } T^{n+1}(A) \longrightarrow T^{n+1}(I_A \oplus I_B) = T^{n+1}(I_A) \oplus T^{n+1}(I_B)$$

$$(T^{n+1}\alpha,\ T^{n+1}\beta \cdot T^{n+1}(g)) = 0$$

Hence a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "g"] & I_A \oplus I_B \arrow[d, "pr_2"] \\
0 \arrow[r] & B \arrow[r] & I_B
\end{tikzcd}$$

which induces $f^{n+1}(g) : T^{n+1}(A) \longrightarrow T^{n+1}(B)$

The commutativity of the left square below is clear. Since the biggest possible square of the diag below follows from the def of $f^{n+1}(A)$, $f^{n+1}(B)$ and the well definedness. The commutativity of the right square also follows

$$\begin{tikzcd}
\operatorname{coker}(T^n(I_A \oplus I_B) \to T^n(A')) = T^{n+1}(A) \arrow[r, "f^{n+1}(A)"] \arrow[d, "\text{coming from } *"'] \arrow[d, phantom, shift left=8ex, "\circlearrowleft"] \arrow[d, shift left=16ex, "T^{n+1}(g)"] & S^{n+1}(A) \arrow[d, "S^{n+1}(g)"] \\
\operatorname{coker}(T^n(I_B) \to T^n(B')) = T^{n+1}(B) \arrow[r, "f^{n+1}(B)"'] & S^{n+1}(B)
\end{tikzcd}$$

Thm. Let $\mathcal{A}$ be an abelian category with enough injectives, $F : \mathcal{A} \to \mathcal{B}$ additive, left exact functor. Then there exists a unique universal δ functor $\{R^iF\}_{i \in \mathbb{N}}$ such that $R^0F = F$.

R^iF is called the i-th (right) derived functor of F

Pf. For $A \in \mathcal{A}$, choose an injective resolution

$$A \to I^\bullet, \quad \text{define } R^iF(A) = H^i(F(I^\bullet))$$

- Since any two injective resolutions are homotopic, different choices of resolutions $I^\bullet$ give isom $\{R^iF(A)\}_{i \in \mathbb{N}}$.

- $R^0F(A) = A$: F is left exact so
$0 \to F(A) \to F(I^0) \to F(I^1)$ is exact
$\Rightarrow H^0(F(I^\bullet)) \xleftarrow{\sim} F(A)$.

- Universality: follows from the obs.

__Prop.__ If $I \in \mathcal{A}$ is an injective object. Then $R^iF(I) = 0, \forall i > 0$.

__Pf.__ Take the injective resolution
$I \to J^\bullet$ where $J^0 = I$, $J^i = 0$ $\forall i > 0$. $\square$.

__Rmk.__ We can play the same game for right exact functors when there are enough projectives.

End of 06.12.24 lecture

__Def/ Notation:__ $(X, \mathcal{O}_X)$ ringed space, $R^i\Gamma(X, -) =: H^i(X, -)$

- For a topological space T, we can consider the category of sheaves of abelian groups and define $H^i(T, -)$
$:= R^i\Gamma(-)$

note a sheaf of abelian gps can be thought of as a sheaf of module over the sheaf of rings $\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is the sheafification of the constant sheaf $U \to \mathbb{Z}$.

- $F = Hom_{\mathcal{O}_X}(\mathcal{F}, -)$ $R^iF(-) =: Ext^i_{\mathcal{O}_X}(\mathcal{F}, -)$.

__Def.__ $F: \mathcal{A} \to \mathcal{B}$ additive left exact. $J \in \mathcal{A}$ is called F acyclic if $R^iF(J) = 0$ $\forall i > 0$.

__Prop.__ $F: \mathcal{A} \to \mathcal{B}$ (additive) left exact functor. For $A \in \mathcal{A}$, suppose there is a resolution of A by F acyclic objects, i.e. $\exists$ complex

$J^0 \to J^1 \to \cdots \in \mathcal{A}$ such that each

J^i is acyclic and $H^0(J^\bullet) \cong A$ and $H^i(J^\bullet) = 0$ $\forall i > 0$.

Then $R^iF(A) \xrightarrow{\sim} H^i(F(J^\bullet))$

__Pf:__ Given an injective resolution $A \to I^\bullet$, we have a map of complexes

$$\begin{array}{ccccccccc}
0 \to & A & \to & I^0 & \to & I^1 & \to & \cdots \\
& \| \, id & & \vdots & & \downarrow & & \\
0 \to & A & \to & J^0 & \to & J^1 & \to & \cdots
\end{array}$$

This induces a map $R^iF(A) = H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$

This map is an isom $\forall i$, but we don't verify that.
Instead we show $R^iF(A) \xrightarrow{\sim} H^i(F(J^\bullet))$ abstractly.

- For $i=0$, $R^0F(A) = A \xrightarrow{\sim} H^0(F(J^\bullet)) \xrightarrow{\sim} A$ as F is left exact.
- We induce on i. Suppose we have isom $\forall A \in \mathcal{A}$, and $i \le n$

Let $A' = \operatorname{coker}(A \to J^0)$, Then $J^\bullet$ gives a resolution $0 \to A' \to J^1 \to J^2 \to \cdots$

The exact seq $0 \to A \to J^0 \to A' \to 0$
gives

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(J^0) \arrow[r] \arrow[d, equal] & F(A') \arrow[r] \arrow[d, equal] & R^1F(A) \arrow[r] \arrow[d, "\wr"] & 0 \arrow[r] & R^1F(A') \arrow[r] & R^2F(A) \arrow[r] & 0 \\
& & F(J^0) \arrow[r] & \ker(F(J^1) \arrow[r] \arrow[d] & H^1(F(J^\bullet)) & & & & \\
& & & F(J^2)) & & & & &
\end{tikzcd}$$

$\Rightarrow$ $R^1F(A) \xrightarrow{\sim} H^1(F(J^\bullet))$

and $R^iF(A') \xrightarrow{\sim} R^{i+1}F(A)$ $\forall i \ge 1$

Since

$$\begin{tikzcd}
R^iF(A') \arrow[r, "\sim"] \arrow[d, "\wr"] & H^i(J^\bullet[1]) \arrow[d, "\wr"] \\
R^{i+1}F(A) \arrow[r, "\sim"] & H^{i+1}(J^\bullet)
\end{tikzcd}$$

for $i \le n$

we are done. ∎

<u>Def:</u> A sheaf $\mathcal{F}$ on a topological space is called flasque if for any two open's in X U, V s.t $U \subseteq V$, $\mathcal{F}(V) \to \mathcal{F}(U)$ is surj.

<u>Prop:</u> $(X, \mathcal{O}_X)$ ringed space. Every injective $\mathcal{O}_X$ mod is flasque

<u>Pf</u> For every open $U \subseteq X$, define $(j_U)_!(\mathcal{F}$

to be the sheafification of
$V \longmapsto 0$ if $V \not\subseteq U$
$V \longmapsto \mathcal{F}(V)$ if $V \subseteq U$

Given $V \subseteq U$ opens

Have an injection $0 \to (j_V)_! \mathcal{O}_X \to (j_U)_! \mathcal{O}_X$

Since $\mathcal{I}$ is injective, $\mathrm{Hom}(-, \mathcal{I})$ gives a surjection

$$0 \leftarrow \mathrm{Hom}_{\mathcal{O}_X}((j_V)_! \mathcal{O}_X, \ ) \leftarrow \mathrm{Hom}_{\mathcal{O}_X}((j_U)_! , \ )$$

$$0 \leftarrow \mathcal{I}(V) \longleftarrow \mathcal{I}(U)$$

Prop: $(X, \mathcal{O}_X)$ ringed space. For an exact seq $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathrm{Mod}_{\mathcal{O}_X}$

if $\mathcal{F}'$ is flasque.

(i) Then $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ is exact

(ii) For a flasque sheaf of abelian groups $\mathcal{G}$, $H^i(X, \mathcal{G}) = 0 \ \forall i > 0$

Pf. Given $s \in \Gamma(X, \mathcal{F}'')$, by Zorn's lemma, choose a maximal elt in the non-empty set

$$\{ (U, t) \mid \emptyset \neq U \underset{\text{open}}{\subseteq} X, \ t \in \mathcal{F}(U) \text{ s.t. } t \text{ maps to } s|_U \},$$

call it (U_s, t_s). If $U_s \neq X$, pick $x \in X \setminus U_s$

$\exists$ a open nbhd V of x and $t_V \in \mathcal{F}(V)$ s.t. t_V maps to $s|_V$.

$$t_V|_{U_s \cap V} - t_s|_{U_s \cap V} \in \mathcal{F}'(U_s \cap V).$$

Since $\mathcal{F}'$ is flasque, $\exists \ \tilde{t} \in \mathcal{F}'(X)$ mapping to the difference above. So $t_V - \tilde{t}|_V$ and t_s agree on $U_s \cap V$. Then $\exists! \ t' \in \mathcal{F}(V \cup U_s)$ s.t. $t'|_{U_s} = t_s$. So $(V \cup U_s, t')$ is bigger than (U_s, t_s) contradicting the maximality.

Thus $U_s = X$.

(ii) Choose an injection $0 \to \mathcal{G} \to \mathcal{I}$ where $\mathcal{I}$ is an injective $\mathcal{O}_X$-mod.

Show that I/g is flasque by considering the diag and that $I, g|_U, I|_U$ are flasque

$$\begin{array}{ccccccccc} 0 & \to & g(X) & \to & I(X) & \to & I/g(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & g(U) & \to & I(U) & \to & I/g(U) & \to & 0 \end{array}$$

Long exact sequence of sheaf cohomology implies

$$0 \to H^0(X,g) \to H^0(X;I) \to H^0(X,I/g) \to H^1(X,g) \to \overset{0}{\overset{\shortparallel}{H^1(X,I)}} \to H^1(X,I/g) \to H^2(X,g) \to 0$$

Since $H^1(X,I)=0$, $(i) \Rightarrow H^1(X,g)=0$

Moreover we have $H^{i+1}(X,g) \simeq H^i(X,I/g)$ $\forall i>0$

By induction show that $H^i(X,g)=0$ $\forall\, i>0$.

<u>Proof</u>: $(X,\mathcal{O}_X)$ ringed space, $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}_X}$.

(i) Given a flasque resolution $\mathcal{F} \to J^\bullet$ i.e each J^i is flasque, $\mathcal{F} \simeq H^0(J^\bullet)$, $H^i(J^\bullet)=0$ $\forall\, i>0$

The i-th cohomology of $\Gamma(X,J^\bullet)$ computes $H^i(X,\mathcal{F})$

(ii) T.F.D.C $\forall\, i \geq 0$

$$\begin{array}{ccl} \text{sheaves of abelian groups} = \mathrm{Mod}_{\underline{\mathbb{Z}}} & \xrightarrow{H^i(X,-)} & \\ \uparrow & & \\ \mathrm{Mod}_{\mathcal{O}_X} & \xrightarrow[H^i(X,-)]{} & \text{abelian groups} \end{array}$$

i.e for an $\mathcal{O}_X$-mod $\mathcal{F}$, the sheaf cohomology of $\mathcal{F}$ obtained by resolving by injective sheaves of abelian gps and injective sheaves of $\mathcal{O}_X$-mods are the same.

<u>Thm</u>: ① A noeth ring, I be an injective mod. Then $\tilde{I}$ is a flasque sheaf.

② On a noeth scheme any q-coh $\mathcal{O}_X$-mod can be resolved by a complex of q.coh flasque $\mathcal{O}_X$-mods.

<u>Pf.</u> <u>Step 1</u>: For an ideal $J \subseteq A$. Set
$$\Gamma_J(I) = \{x \in I \mid J^n \cdot x = 0 \text{ for some } n \in \mathbb{N}\}$$

Pf. Step 1: For an ideal $J \subseteq A$. Set

$$\Gamma_J(I) = \{x \in I \mid J^n . x = 0 \text{ for some } n \in \mathbb{N}\}$$

$$= \{x \in I \mid \tfrac{x}{1} \in I_p \text{ is zero if } p \notin V(J)\}.$$

So $\Gamma_J(I)$ only depends on $V(J)$ and I.
So $\Gamma_J(I) = \Gamma_{V(J)}(I)$

Claim: $\Gamma_J(I)$ is an injective A-mod. (Lemma 3.2, Hart)

Step 2: For any $f \in A$, the map $I \to I_f$ is surjective. (Lemma 3.3, Hart)

Noeth induction: T noeth top space (i.e every decreasing chain of closed sets stabilize). Let P be a property of closed subset such that for every closed $Z \subseteq T$ if P holds for every proper closed subset of Z, P holds for T. Then P holds for X. [Note by our assumption P holds for $\emptyset$]

Pf of noeth induction:
Consider the collection of closed subset on which P fails. If this set is non-empty there must be a smallest elt. But our assumption contradicts that. $\square$

X scheme, $\mathcal{F} \in$ Mod$_{\mathcal{O}_X}$
Define $\operatorname{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$
For $s \in \mathcal{F}(U)$, $\operatorname{supp}(s) = \{x \in U \mid s_x \neq 0\}$

$X = \operatorname{Spec}(A)$. We say that a closed subset Z has property P if for every injective A-module M with $\operatorname{Supp}(M) \subseteq Z$ $\tilde{M}$ is flasque.

Let Z be a closed subset s.t every proper closed subset has property P. Take an injective mod M s.t $\operatorname{Supp} \tilde{M} \subseteq Z$.

Given $U \underset{\text{open}}{\subseteq} X$, if $U \cap Z = \emptyset$,

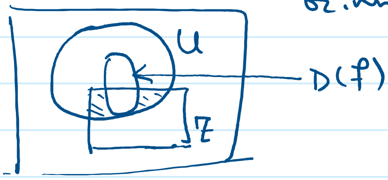
$\tilde{M}(X) \longrightarrow \tilde{M}(U) = 0$ is surjective.

Assume $U \cap Z \neq \emptyset$ choose $f \in A$ s.t $D(f) \subseteq U$ and $D(f) \cap Z \neq \emptyset$. $Z' = X - D(f) = V(f)$

Define $M' = \{x \in M \mid f^n . x = 0 \text{ for some } n \text{ depending on } x\}$

Note $\tilde{M}'(U) = \{s \in \tilde{M}(U) \mid \operatorname{supp}(s) \subseteq U \cap Z'\}$

(4) Since $\operatorname{supp}(M') \subseteq V(f) \cap \operatorname{supp}(M) \subseteq V(f) \cap Z \subsetneq Z$, $\tilde{M}'$ is flasque by our induction hypothesis



Given $s \in \Gamma(U, \tilde{M})$, $\exists\ s' \in \Gamma(X, \tilde{M})$ s.t $s'|_{D(f)} = s|_{D(f)}$

Thus $s - s'|_U \in \Gamma(U, \tilde{M})$ has support in $V(f) \cap Z$

Thus $s - s'|_U \in \Gamma(U, \tilde{M}') \subseteq \Gamma(U, \tilde{M})$. Since $\tilde{M}'$ is flasque

[By our induction hypo, since $\overline{\operatorname{supp}\ (\Gamma_{Z'}(M))} \subsetneq Z$, $\widetilde{\Gamma_{Z'}(M)}$ is flasque] Choose $t \in \tilde{M}'(X) = M' \subseteq M$

s.t $t|_U = s - s'|_U$

Then $(s' + t)|_U = s$, $s' + t \in \Gamma(X, \tilde{M}) = M$.

Prop: X be a noetherian scheme. Every q.coh $\mathcal{O}_X$ mod $\mathcal{F}$ admits a resolution by q coh flasque sheaves.

Pf. Given $\mathcal{F} \in \operatorname{Qcoh}(X)$, enough to produce an injection

$0 \to \mathcal{F} \to I$, where I is a q.coh flasque $\mathcal{O}_X$-mod.

Choose an affine open covering $X = \bigcup_{i=1}^{n} U_i$.

For each i, choose an injection $0 \to \mathcal{F}|_{U_i} \to I_i$ where I_i is an injective $\mathcal{O}_X(U_i)$-mod. Denote the immersion $U_j \to X$ by i_j

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[rr, bend right] & \bigoplus_{j=1}^{n} (i_j)_*(\mathcal{F}|_{U_j}) \arrow[r] & \bigoplus_{j=1}^{n} (i_j)_*(\tilde{I}_j) \end{tikzcd}$$

Then get an injection as above, where $f(\mathcal{F}|_V) \longrightarrow (f|_{V \cap U_j})$

Claim: $\bigoplus_{j=1}^{n} (i_j)_*(\tilde{I}_j)$ is flasque.

Pf: $\tilde{I}_j$ is flasque on U_j, so $(i_j)_*(\tilde{I}_j)$ is flasque, so is the direct sum.

Pf. $\tilde{I}_j$ is flasque on U_j, so $(i_j)_*(\tilde{I}_j)$ is flasque, so is the direct sum.

Thm. Let X be a noeth affine scheme. $\mathcal{F} \in Qcoh(X)$.

$$H^i(X, \mathcal{F}) = 0 \quad \forall \quad i > 0.$$

Rmk. THE ABOVE THM IS TRUE WITHOUT ANY NOETH HYPOTHESIS.

Pf. Let $X = \operatorname{Spec}(A)$, $\mathcal{F} \cong \tilde{M}$ for some $M \in Mod_A$.

Choose an injective resolution $0 \to M \to I^0 \to I^1 \to$ in Mod_A.

Then have a flasque resolution in $Mod_{\mathcal{O}_X}$

$$0 \to \tilde{M} \to \tilde{I}^0 \to \tilde{I}^1 \to \cdots$$

Since flasque sheaves are $\Gamma(X, -)$ acyclic

$$H^i(X, \tilde{M}) = H^i(\Gamma(\tilde{I}^\bullet))$$

$$= H^i(I^0 \to I^1 \to \cdots \to I^i \to I^{i+1} \to \cdots)$$

$\because I^j$ q.coh, $\tilde{I}^j(X) \cong I^j$

$$= 0 \text{ for } i > 0.$$

Thm. Let X be a noeth scheme. T.F.A.E.

① X is affine

② $\forall$ q.coh $\mathcal{O}_X$-mod $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ $\forall i > 0$.

③ For all q.coh ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, $H^i(X, \mathcal{J}) = 0$ $\forall i > 0$.

Pf. Hart Thm 3.7.